Pre-class Warm-up!!!

1. What surface does the function

$\Phi(u,v) = (u\cos v, u\sin v, v)$, $0 \le u \le 1$, $0 \le v \le 2\pi$

parametrize?

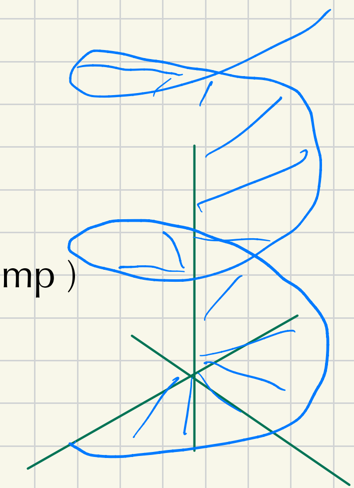
a. a cone

b. a cylinder

✓ c. a helicoid (a spiral ramp)

d. a sphere

e. None of the above.



2. Which integral computes its area?

a. $\int_0^{2\pi}\int_0^1 \|\Phi\| \, du \, dv$

b. $\int_0^{2\pi}\int_0^1 (T_u \times T_v) \cdot dS$

✓ c. $\int_0^{2\pi}\int_0^1 \|T_u \times T_v\| \, du \, dv$

d. $\int_0^{2\pi}\int_0^1 du \, dv$

e. None of the above

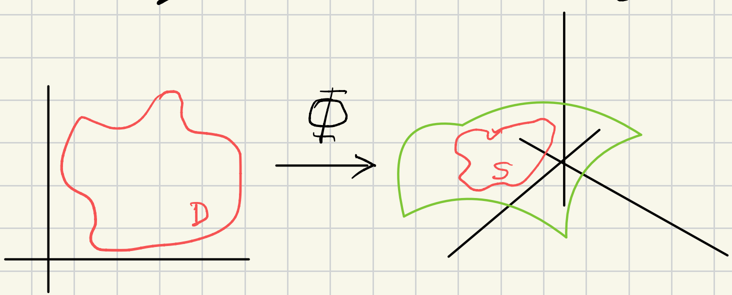
Section 7.5 Surface integrals of functions.

We learn: $f: \mathbb{R}^3 \to \mathbb{R}$

- the same as in section 7.4 (which was about finding surface area) made a bit more complicated by integrating a scalar-valued function over some surface.
- We don't really need page 396: Integrals over graphs.

Recall: the area of a surface S parametrized by a function (x,y,z) =Phi(u,v) is given by

$$A = \iint_D \|T_u \times T_v\| \, du\, dv = \iint_S dS$$



$$T_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \quad T_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

New formula. We suppose we have a function f : R^3 -> R .

$$\iint_S f \, dS = \iint_D f(\Phi(u,v)) \, \|T_u \times T_v\| \, du\, dv$$

Interpretation:

1. We now do integrals over curvy surfaces; before, we integrated over flat parts of $\mathbb{R}^2$.

2. If the surface is a film with surface density $f(x,y,z)$, the mass of the film is

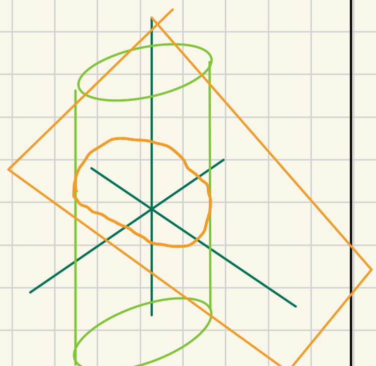
$$\iint_S f \, dS$$

Like question 6:

Evaluate $\displaystyle\iint_S xz\,dS$

where S is the part of the plane z = 1 + 2x - 3y that lies inside the cylinder x^2 + y^2 = 4.

Solution.



Step. Parametrize

$$\Phi(u,v) = (u, v, 1+2u-3v)$$

$$u^2+v^2 \le 4$$

Step2 $T_u = (1,0,2)$ $T_v = (0,1,-3)$

$$T_u \times T_v = (-2,3,1) \qquad \|T_u \times T_v\| = \sqrt{4+9+1} = \sqrt{14}$$

Step3 The integral is $\displaystyle\iint_{u^2+v^2\le 4} u(1+2u-3v)\sqrt{14}\,du\,dv$

Step4 Evaluate by using polar coordinates

$$\int_0^{2\pi}\int_0^2 r\cos\theta(1+2r\cos\theta-3r\sin\theta)\sqrt{14}\,r\,dr\,d\theta$$

$$= \ldots$$

What about $\displaystyle\iint_S x\,dS$?

We would get 0 because x takes equal and opposite values on two halves of $x^2+y^2\le 4$.

Example: A unit sphere is made of material with surface density z^2 gm/cm^2 at point (x,y,z). Find the mass of the sphere.

Solution: Evaluate $\iint_{\text{sphere}} z^2 \, dS$

Phi(u,v) = (sin v cos u, sin v sin u, cos v)
$0 \leq u \leq 2\pi, \ 0 \leq v \leq \pi$

T_u = (- sin v sin u, sin v cos u, 0)
T_v = (cos v cos u, cos v sin u, -sin v)

T_u x T_v = (-sin^2 v cos u, -sin^2 v sin u, -sin v cos v)

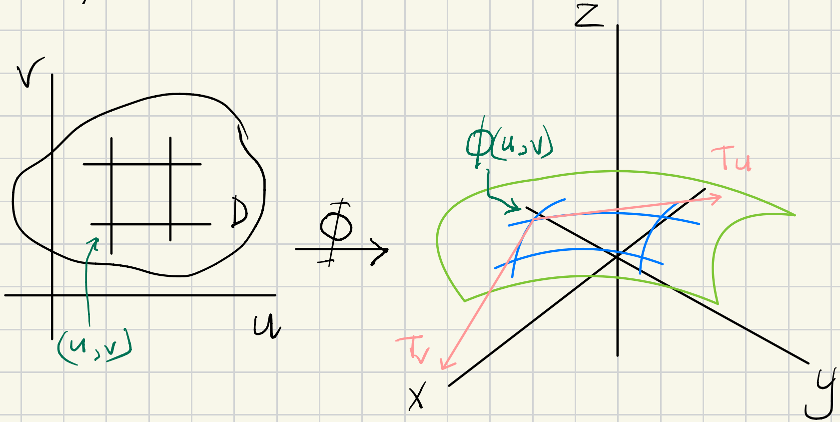
|| T_u x T_v || = $\sqrt{\text{sin^4 v + sin^2 v cos^2 v}}$ = sin v

Mass = $\int_0^{\pi} \int_0^{2\pi} (\cos v)^2 \sin v \, du \, dv = \ldots$

Question: what is z when expressed in (u,v)-coordinates?

a. sin v
b. cos v ✓
c. tan v
d. sin u
e. cos u

Why it works:



The scaling factor by which the parallelogram area in the (x,y,z) space is multiplied from the the little square in the (u,v,) plane, is

|| T_u x T_v ||